A CONSTRUCTION OF CHERN CLASSES OF PARABOLIC VECTOR BUNDLES

INDRANIL BISWAS AND AJNEET DHILLON

Abstract. Given a parabolic vector bundle, we construct for it of projectivization and tautological line bundle. These are analogs of the projectivization and tautological line bundle for an usual vector bundle. Using these we give a construction of the parabolic Chern classes.

arXiv:1209.3160v1 [math.AG] 14 Sep 2012

1. Introduction

Parabolic vector bundles on a smooth complex projective curve were introduced in [Se]. In [MY], Maruyama and Yokogawa introduced parabolic vector bundles on higher dimensional complex projective varieties. The notion of Chern classes of a vector bundle extends to the context of parabolic vector bundles [Bi2], [IS], [Ta].

Take a vector bundle V of rank r on a variety Z. Let $\psi : \mathbb{P}(V) \longrightarrow Z$ be the projective bundle parametrizing hyperplanes in the fibers of the vector bundle V. The tautological line bundle on $\mathbb{P}(V)$ will be denoted by $\mathcal{O}_{\mathbb{P}(V)}(1)$. One of the standard ways of constructing Chern classes of V is to use the identity

$$\sum_{i=0}^{r}(-1)^i c_1(\mathcal{O}_{\mathbb{P}(V)}(1))^{r-i}\psi^* c_i(V) \,=\, 0$$

with $c_0(V) \,=\, 1$.

Our aim here is to give a construction of Chern classes of parabolic vector bundles along this line (see Theorem 3.3).

N. Borne showed that parabolic vector bundles can be understood as vector bundles on root-stacks [Bo1], [Bo2]. In terms of this correspondence, the i-th parabolic Chern class of a parabolic vector bundle is the usual i-th Chern class of the corresponding vector bundle on root-stack. It should be mentioned that this elegant correspondence in [Bo1], [Bo2] between parabolic vector bundles and vector bundles on root-stacks is turning out to be very useful (see, for example, [BD] for application of this correspondence).

2. Preliminaries

2.1. **Parabolic vector bundles.** Let X be an irreducible smooth projective variety defined over $\mathbb{C}$. Let $D \subset X$ be an effective reduced divisor satisfying the condition that each irreducible component of D is smooth, and the irreducible components of D intersect

2000 *Mathematics Subject Classification.* 14F05, 14C15.

Key words and phrases. Parabolic bundle, Chern class, equivariant Chow group, ramified bundle.

transversally; divisors satisfying these conditions are called simple normal crossing ones. Let

$$D = \sum_{i=1}^{\ell} D_i \tag{2.1}$$

be the decomposition of D into irreducible components.

Let E_0 be an algebraic vector bundle over X. For each $i \in [1\,,\ell]$, let

$$E_0|_{D_i} = F_1^i \supsetneq F_2^i \supsetneq \cdots \supsetneq F_{m_i}^i \supsetneq F_{m_i+1}^i = 0 \tag{2.2}$$

be a filtration by subbundles of the restriction of E_0 to D_i.

A *quasiparabolic* structure on E_0 over D is a filtration as above of each $E_0|_{D_i}$ such that the system of filtrations is locally abelian (see [Bo2, p. 157, Definition 2.4.19] for the definition of a locally abelian structure).

For a quasiparabolic structure as above, *parabolic weights* are a collection of rational numbers

$$0 \leq \lambda_1^i < \lambda_2^i < \lambda_3^i < \cdots < \lambda_{m_i}^i < 1\,,$$

where $i \in [1\,,\ell]$. The parabolic weight λ_j^i corresponds to the subbundle F_j^i in (2.2). A *parabolic structure* on E_0 is a quasiparabolic structure on E_0 (defined as above) equipped with parabolic weights. A vector bundle over X equipped with a parabolic structure on it is also called a *parabolic vector bundle*. (See [MY], [Se].) For notational convenience, a parabolic vector bundle defined as above will be denoted by E_*.

The divisor D is called the *parabolic divisor* for E_*. We fix D once and for all. So the parabolic divisor of all parabolic vector bundles on X will be D.

The definitions of direct sum, tensor product and dual of vector bundles extend naturally to parabolic vector bundles; similarly, symmetric and exterior powers of parabolic vector bundles are also constructed (see [MY], [Bi3], [Yo]).

2.2. Ramified principal bundles.

The complement of D in X will be denoted by $X-D$.

Let

$$\varphi : E_{\mathrm{GL}(r,\mathbb{C})} \longrightarrow X$$

be a ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundle with ramification over D (see [BBN], [Bi3], [Bi4]). We briefly recall its defining properties. The total space $E_{\mathrm{GL}(r,\mathbb{C})}$ is a smooth complex variety equipped with an algebraic right action of $\mathrm{GL}(r,\mathbb{C})$

$$f : E_{\mathrm{GL}(r,\mathbb{C})} \times \mathrm{GL}(r,\mathbb{C}) \longrightarrow E_{\mathrm{GL}(r,\mathbb{C})}\,, \tag{2.3}$$

and the following conditions hold:

(1) $\varphi\circ f = \varphi\circ p_1$, where p_1 is the natural projection of $E_{\mathrm{GL}(r,\mathbb{C})}\times\mathrm{GL}(r,\mathbb{C})$ to $E_{\mathrm{GL}(r,\mathbb{C})}$,
(2) for each point $x \in X$, the action of $\mathrm{GL}(r,\mathbb{C})$ on the reduced fiber $\varphi^{-1}(x)_{\mathrm{red}}$ is transitive,
(3) the restriction of φ to $\varphi^{-1}(X-D)$ makes $\varphi^{-1}(X-D)$ a principal $\mathrm{GL}(r,\mathbb{C})$–bundle over $X-D$,

(4) for each irreducible component $D_i \subset D$, the reduced inverse image $\varphi^{-1}(D_i)_{\rm red}$ is a smooth divisor and

$$\widehat{D} := \sum_{i=1}^{\ell} \varphi^{-1}(D_i)_{\rm red}$$

is a normal crossing divisor on $E_{\mathrm{GL}(r,\mathbb{C})}$, and

(5) for any point x of D, and any point $z \in \varphi^{-1}(x)$, the isotropy group

$$G_z \subset \mathrm{GL}(r,\mathbb{C})\,, \tag{2.4}$$

for the action of $\mathrm{GL}(r,\mathbb{C})$ on $E_{\mathrm{GL}(r,\mathbb{C})}$, is a finite group, and if x is a smooth point of D, then the natural action of G_z on the quotient line $T_zE_{\mathrm{GL}(r,\mathbb{C})}/T_z\varphi^{-1}(D)_{\rm red}$ is faithful.

Let

$$D^{\rm sm} \subset D$$

be the smooth locus of the divisor. Take any point $x \in D^{\rm sm}$, and let $z \in \varphi^{-1}(x)$ be any point. Since G_z acts faithfully on the line $T_zE_{\mathrm{GL}(r,\mathbb{C})}/T_z\varphi^{-1}(D)_{\rm red}$, it follows that G_z is a cyclic group. Take any $z' \in E_{\mathrm{GL}(r,\mathbb{C})}$ such that $\varphi(z') = \varphi(z)$. There is an element $g \in \mathrm{GL}(r,\mathbb{C})$ such that $f(z\,,g) = z'$. Therefore, the subgroup G_z is conjugate to the subgroup $G_{z'}$; more precisely, we have $g^{-1}G_zg = G_{z'}$. In particular, G_z is isomorphic to $G_{z'}$.

There is a natural bijective correspondence between the ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundles with ramification over D and the parabolic vector bundles of rank r (see [BBN], [Bi3]). We first describe the steps in the construction of a ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundle from a parabolic vector bundle of rank r:

- Given a parabolic vector bundle E_* of rank r on X, there is a Galois covering

$$\gamma : Y \longrightarrow X\,, \tag{2.5}$$

  where Y is an irreducible smooth projective variety, and a $\mathrm{Gal}(\gamma)$–linearized vector bundle F on Y [Bi1], [Bo1], [Bo2]. Let $F_{\mathrm{GL}(r,\mathbb{C})}$ be the principal $\mathrm{GL}(r,\mathbb{C})$–bundle on Y defined by F. We recall that $F_{\mathrm{GL}(r,\mathbb{C})}$ is the space of all linear isomorphisms from $\mathbb{C}^r$ to the fibers of F.
- The linearization action of $\mathrm{Gal}(\gamma)$ on F produces an action of $\mathrm{Gal}(\gamma)$ on $F_{\mathrm{GL}(r,\mathbb{C})}$. This action of $\mathrm{Gal}(\gamma)$ on $F_{\mathrm{GL}(r,\mathbb{C})}$ commutes with the action of $\mathrm{GL}(r,\mathbb{C})$ on $F_{\mathrm{GL}(r,\mathbb{C})}$ because it is given by a linearization action on F.
- The quotient

$$\mathrm{Gal}(\gamma)\backslash F_{\mathrm{GL}(r,\mathbb{C})} \longrightarrow \mathrm{Gal}(\gamma)\backslash Y = X$$

  is a ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundle.

It is starightforward to check that $\mathrm{Gal}(\gamma)\backslash F_{\mathrm{GL}(r,\mathbb{C})}$ is a ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundle over X.

We will now describe the construction of a parabolic vector bundles of rank r from a ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundle.

Let

$$\varphi : F_{\mathrm{GL}(r,\mathbb{C})} \longrightarrow X$$

be a ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundle. Let f be as in (2.3). Consider the trivial vector bundle

$$W := F_{\mathrm{GL}(r,\mathbb{C})} \times \mathbb{C}^r \longrightarrow F_{\mathrm{GL}(r,\mathbb{C})}.$$

The group $\mathrm{GL}(r,\mathbb{C})$ acts on $F_{\mathrm{GL}(r,\mathbb{C})} \times \mathbb{C}^r$ as follows: the action of any $g \in \mathrm{GL}(r,\mathbb{C})$ sends any $(z, v) \in F_{\mathrm{GL}(r,\mathbb{C})} \times \mathbb{C}^r$ to $(f(z,g), g^{-1}(v))$. Note that this action on W is a lift of the action of $\mathrm{GL}(r,\mathbb{C})$ on $F_{\mathrm{GL}(r,\mathbb{C})}$ defined by f. This action of $\mathrm{GL}(r,\mathbb{C})$ on W produces an action of $\mathrm{GL}(r,\mathbb{C})$ on the quasicoherent sheaf $\varphi_* W$ on X. Note that this action commutes with the trivial action of $\mathrm{GL}(r,\mathbb{C})$ on $X = F_{\mathrm{GL}(r,\mathbb{C})}/\mathrm{GL}(r,\mathbb{C})$.

The vector bundle underlying the parabolic vector bundle corresponding to $F_{\mathrm{GL}(r,\mathbb{C})}$ is

$$E_0 := (\varphi_* W)^{\mathrm{GL}(r,\mathbb{C})} \subset \varphi_* W.$$

Here $(\varphi_* W)^{\mathrm{GL}(r,\mathbb{C})}$ denotes the sheaf of invariants; from the given conditions on $F_{\mathrm{GL}(r,\mathbb{C})}$ it follows that E_0 is a locally free coherent sheaf. We will construct a parabolic structure on E_0.

For any $i \in [1, \ell]$, the reduced divisor $\varphi^{-1}(D_i)_{\mathrm{red}} \subset F_{\mathrm{GL}(r,\mathbb{C})}$ is preserved by the action of $\mathrm{GL}(r,\mathbb{C})$ on $F_{\mathrm{GL}(r,\mathbb{C})}$. Therefore, the line bundle

$$\mathcal{O}_{F_{\mathrm{GL}(r,\mathbb{C})}}(\varphi^{-1}(D_i)_{\mathrm{red}}) \longrightarrow F_{\mathrm{GL}(r,\mathbb{C})}$$

is equipped with a lift of the action of $\mathrm{GL}(r,\mathbb{C})$ on $F_{\mathrm{GL}(r,\mathbb{C})}$. For each $n \in \mathbb{Z}$, the action on $\mathrm{GL}(r,\mathbb{C})$ on $\mathcal{O}_{F_{\mathrm{GL}(r,\mathbb{C})}}(\varphi^{-1}(D_i)_{\mathrm{red}})$ produces an action of $\mathrm{GL}(r,\mathbb{C})$ on the line bundle $\mathcal{O}_{F_{\mathrm{GL}(r,\mathbb{C})}}(n \cdot \varphi^{-1}(D_i)_{\mathrm{red}})$.

For any $i \in [1, \ell]$, take any point $x_i \in D^{\mathrm{sm}} \bigcap D_i$, where D^{sm} as before is the smooth locus of D. Recall that the order of the cyclic isotropy subgroup $G_z \in \mathrm{GL}(r,\mathbb{C})$, where $z \in \varphi^{-1}(x_i)$, is independent of the choices of both x_i and z. Let n_i be the order of G_z, where z is as above.

For any real number λ, by $[\lambda]$ we will denote the integral part of λ. So, $[\lambda] \in \mathbb{Z}$, and $0 \leq \lambda - [\lambda] < 1$.

For any $t \in \mathbb{R}$, consider the vector bundle

$$W_t := W \otimes \mathcal{O}_{F_{\mathrm{GL}(r,\mathbb{C})}}(\sum_{i=1}^{\ell} [-tn_i] \cdot \varphi^{-1}(D_i)_{\mathrm{red}}) \longrightarrow F_{\mathrm{GL}(r,\mathbb{C})},$$

where n_i is defined above. The actions of $\mathrm{GL}(r,\mathbb{C})$ on W and $\mathcal{O}_{F_{\mathrm{GL}(r,\mathbb{C})}}(\varphi^{-1}(D_i)_{\mathrm{red}})$ together produce an action of $\mathrm{GL}(r,\mathbb{C})$ on the vector bundle W_t defined above. This action of $\mathrm{GL}(r,\mathbb{C})$ on W_t lifts the action of $\mathrm{GL}(r,\mathbb{C})$ on $F_{\mathrm{GL}(r,\mathbb{C})}$.

Let

$$E_t := (\varphi_* W_t)^{\mathrm{GL}(r,\mathbb{C})} \subset \varphi_* W_t$$

be the invariant direct image. This E_t is a locally free coherent sheaf on X.

This filtration of coherent sheaves $\{E_t\}_{t \in \mathbb{R}}$ defines a parabolic vector bundle on X with E_0 as thr underlying vector bundle (see [MY] for the description of a parabolic vector

bundles as a filtration of sheaves). The proof of it similar to the proofs in [Bi1], [Bo1], [Bo2].

The above construction of a parabolic vector bundle of rank r from a ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundle is the inverse of the earlier construction of a ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundle from a parabolic vector bundle.

We note that the above construction of a parabolic vector bundle of rank r from a ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundle coincides with the following construction (however we do not need this here for our purpose).

As before, let $F_{\mathrm{GL}(r,\mathbb{C})} \longrightarrow X$ be a ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundle. Then there is a finite (ramified) Galois covering

$$\gamma : Y \longrightarrow X$$

such that the normalization $\widetilde{F_{\mathrm{GL}(r,\mathbb{C})} \times_X Y}$ of the fiber product $F_{\mathrm{GL}(r,\mathbb{C})} \times_X Y$ is smooth. The projection $\widetilde{F_{\mathrm{GL}(r,\mathbb{C})} \times_X Y} \longrightarrow Y$ is a principal $\mathrm{GL}(r,\mathbb{C})$–bundle equipped with an action of the Galois group $\Gamma := \mathrm{Gal}(\gamma)$. Let $F_{V_0} := \widetilde{F_{\mathrm{GL}(r,\mathbb{C})} \times_X Y}(V_0)$ be the vector bundle over Y associated to the principal $\mathrm{GL}(r,\mathbb{C})$–bundle $\widetilde{F_{\mathrm{GL}(r,\mathbb{C})} \times_X Y}$ for the standard $\mathrm{GL}(r,\mathbb{C})$–module $V_0 := \mathbb{C}^r$. The action of Γ on Y induces an action of Γ on $\widetilde{F_{\mathrm{GL}(r,\mathbb{C})} \times_X Y}$; this action of Γ on $\widetilde{F_{\mathrm{GL}(r,\mathbb{C})} \times_X Y}$ commutes with the action of $\mathrm{GL}(r,\mathbb{C})$ on $\widetilde{F_{\mathrm{GL}(r,\mathbb{C})} \times_X Y}$. Hence the action of Γ on $\widetilde{F_{\mathrm{GL}(r,\mathbb{C})} \times_X Y}$ induces an action of Γ on the above defined associated bundle F_{V_0} making F_{V_0} a Γ–linearized vector bundle. Let E_* be the parabolic vector bundle of rank r over X associated to this Γ–linearized vector bundle F_{V_0}.

Take an irreducible component D_i of the parabolic divisor D. Consider the parabolic vector bundle E_* constructed above from the ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundle $F_{\mathrm{GL}(r,\mathbb{C})} \longrightarrow X$. A rational number $0 \geq \lambda < 1$ is a parabolic weight for the quasi-parabolic filtration of E_* over D_i if and only if $\exp(2\pi\sqrt{-1}\lambda)$ is an eigenvalue of the isotropy subgroup G_z for a general point z of D_i; if λ is a parabolic weight, then its multiplicity coincides with the multiplicity of the eigenvalue $\exp(2\pi\sqrt{-1}\lambda)$ of G_z.

3. Chern classes of parabolic vector bundles

3.1. **Projective bundle and the tautological line bundle.** Let E_* be a parabolic vector bundle over X of rank r. Let

$$\varphi : E_{\mathrm{GL}(r,\mathbb{C})} \longrightarrow X \tag{3.1}$$

be the corresponding ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundle. Let $\mathbb{P}^{r-1}$ be the projective space parametrizing the hyperplanes in $\mathbb{C}^r$. The standard action of $\mathrm{GL}(r,\mathbb{C})$ on $\mathbb{C}^r$ produces an action of $\mathrm{GL}(r,\mathbb{C})$ on $\mathbb{P}^{r-1}$. Let

$$\rho : \mathrm{GL}(r,\mathbb{C}) \longrightarrow \mathrm{Aut}(\mathbb{P}^{r-1}) \tag{3.2}$$

be the homomorphism defined by this action. Let

$$\mathbb{P}(E_*) = E_{\mathrm{GL}(r,\mathbb{C})}(\mathbb{P}^{r-1}) := E_{\mathrm{GL}(r,\mathbb{C})} \times^{\mathrm{GL}(r,\mathbb{C})} \mathbb{P}^{r-1} \longrightarrow X \tag{3.3}$$

be the associated (ramified) fiber bundle. We note that $\mathbb{P}(E_*)$ is a quotient of $E_{\mathrm{GL}(r,\mathbb{C})} \times \mathbb{P}^{r-1}$; two points $(y_1\,, z_1)$ and $(y_2\,, z_2)$ of $E_{\mathrm{GL}(r,\mathbb{C})} \times \mathbb{P}^{r-1}$ are identified in $\mathbb{P}(E_*)$ if there is an element $g \in \mathrm{GL}(r,\mathbb{C})$ such that $y_2 = y_1 g$ and $z_2 = \rho(g^{-1})(z_1)$, where ρ is the homomorphism in (3.2).

Definition 3.1. We will call $\mathbb{P}(E_*)$ the *projective bundle* associated to the parabolic vector bundle E_*.

Take a point $x \in D$; it need not be a smooth point of D. Take any $z \in \varphi^{-1}(x)$, where φ is the morphism in (3.1). As in (2.4), let $G_z \subset \mathrm{GL}(r,\mathbb{C})$ be the isotropy subgroup of z for the action of $\mathrm{GL}(r,\mathbb{C})$ on $E_{\mathrm{GL}(r,\mathbb{C})}$. We recall that G_z is a finite group. Let n_x be the order of G_z; we note that n_x is independent of the choice of z in $\varphi^{-1}(x)$ because for any other $z' \in \varphi^{-1}(x)$, the two groups $G_{z'}$ and G_z are isomorphic. The number of distinct integers n_x, $x \in D$, is finite. Indeed, this follows immediately from the fact that as x moves over a fixed connected component of D^{sm}, the conjugacy class of the subgroup $G_z \subset \mathrm{GL}(r,\mathbb{C})$, $z \in \varphi^{-1}(x)$, remains unchanged.

Let

$$N(E_*) := \mathrm{l.c.m.}\{n_x\}_{x\in D} \tag{3.4}$$

be the least common multiple of all these integers n_x.

As before, $\mathbb{P}^{r-1}$ is the projective space parametrizing the hyperplanes in $\mathbb{C}^r$. For any point $y \in \mathbb{P}^{r-1}$, let

$$H_y \subset \mathrm{GL}(r,\mathbb{C}) \tag{3.5}$$

be the isotropy subgroup for the action of $\mathrm{GL}(r,\mathbb{C})$ on $\mathbb{P}^{r-1}$ constructed using ρ in (3.2). So H_y is a maximal parabolic subgroup of $\mathrm{GL}(r,\mathbb{C})$. Let $\mathcal{O}_{\mathbb{P}^{r-1}}(1) \longrightarrow \mathbb{P}^{r-1}$ be the tautological quotient line bundle. The group H_y in (3.5) acts on the fiber $\mathcal{O}_{\mathbb{P}^{r-1}}(1)_y$ over the point y.

For points $x \in D$, $z \in \varphi^{-1}(x)$ and $y \in \mathbb{P}^{r-1}$, the group $G_z \bigcap H_y \subset \mathrm{GL}(r,\mathbb{C})$, where G_z and H_y are defined in (2.4) and (3.5) respectively, acts trivially on the fiber $\mathcal{O}_{\mathbb{P}^{r-1}}(n_x)_y$ of the line bundle $\mathcal{O}_{\mathbb{P}^{r-1}}(n_x) := \mathcal{O}_{\mathbb{P}^{r-1}}(1)^{\otimes n_x}$ over y. Indeed, this follows from the fact that n_x is the order of G_z. Therefore, from the definition of $N(E_*)$ in (3.4) it follows immediately that for any $z \in \varphi^{-1}(D)$ and any $y \in \mathbb{P}^{r-1}$, the group $G_z \bigcap H_y \subset \mathrm{GL}(r,\mathbb{C})$ acts trivially on the fiber of the line bundle

$$\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_*)) := \mathcal{O}_{\mathbb{P}^{r-1}}(1)^{\otimes N(E_*)}$$

over the point y.

Consider the action of $\mathrm{GL}(r,\mathbb{C})$ on the total space of the line bundle $\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_*))$ constructed using the standard action of $\mathrm{GL}(r,\mathbb{C})$ on $\mathbb{C}^r$. Let

$$E_{\mathrm{GL}(r,\mathbb{C})}(\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_*))) := E_{\mathrm{GL}(r,\mathbb{C})} \times^{\mathrm{GL}(r,\mathbb{C})} \mathcal{O}_{\mathbb{P}^{r-1}}(N(E_*)) \longrightarrow X$$

be the associated fiber bundle. Since the natural projection

$$\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_*)) \longrightarrow \mathbb{P}^{r-1}$$

intertwines the actions of $\mathrm{GL}(r,\mathbb{C})$ on $\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_*))$ and $\mathbb{P}^{r-1}$, this natural projection produces a projection

$$E_{\mathrm{GL}(r,\mathbb{C})}(\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_*))) \longrightarrow \mathbb{P}(E_*) \tag{3.6}$$

between the associated bundles, where $\mathbb{P}(E_*)$ is constructed in (3.3).

Using the above observation that $G_z \bigcap H_y$ acts trivially on the fiber of $\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_*))$ over y it follows immediately that the projection in (3.6) makes $E_{\mathrm{GL}(r,\mathbb{C})}(\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_*)))$ an algebraic line bundle over the variety $\mathbb{P}(E_*)$.

Definition 3.2. The line bundle $E_{\mathrm{GL}(r,\mathbb{C})}(\mathcal{O}_{\mathbb{P}^{r-1}}(N(E_*))) \longrightarrow \mathbb{P}(E_*)$ will be called the *tautological line bundle;* this tautological line bundle will be denoted by $\mathcal{O}_{\mathbb{P}(E_*)}(1)$.

3.2. **Chern class of the tautological line bundle.** For any nonnegative integer i, define the rational Chow group $\mathrm{CH}^i(X)_{\mathbb{Q}} := \mathrm{CH}^i(X) \bigotimes_{\mathbb{Z}} \mathbb{Q}$.

Let E_* be a parabolic vector bundle over X of rank r. The corresponding ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundle over X will be denoted by $E_{\mathrm{GL}(r,\mathbb{C})}$. Consider $\mathbb{P}(E_*)$ constructed as in (3.3) from $E_{\mathrm{GL}(r,\mathbb{C})}$. Let

$$\psi : \mathbb{P}(E_*) \longrightarrow X$$

be the natural projection. Let $\mathcal{O}_{\mathbb{P}(E_*)}(1)$ be the tautological line bundle over $\mathbb{P}(E_*)$ (see Definition 3.2).

Theorem 3.3. *For each integer $i \in [0\,, r]$, there is a unique element*

$$\widetilde{C}_i(E_*) \in \mathrm{CH}^i(X)_{\mathbb{Q}}$$

such that

$$\sum_{i=0}^{r}(-1)^i c_1(\mathcal{O}_{\mathbb{P}(E_*)}(1))^{r-i}\psi^*\widetilde{C}_i(E_*) = 0 \tag{3.7}$$

with $\widetilde{C}_0(E_*) = 1/N(E_*)^r$, where $N(E_*)$ is the integer in (3.4).

Proof. Let $\gamma : Y \longrightarrow X$ be the covering in (2.5) (recall that it depends on E_*). Let $E' \longrightarrow Y$ be the Γ–linearized vector bundle over Y corresponding to E_*, where $\Gamma = \mathrm{Gal}(\gamma)$ is the Galois group of γ. Let $\mathbb{P}(E')$ be the projective bundle over Y parametrizing the hyperplanes in the fibers of E'. The tautological line bundle over $\mathbb{P}(E')$ will be denoted by $\mathcal{O}_{\mathbb{P}(E')}(1)$.

The action of Γ on E' produces an action of Γ on $\mathbb{P}(E')$ lifting the action of Γ on Y. It can be seen that the variety $\mathbb{P}(E_*)$ in (3.3) is the quotient

$$\Gamma\backslash\mathbb{P}(E') = \mathbb{P}(E_*)\,. \tag{3.8}$$

Indeed, this follows immediately from the fact that $\Gamma\backslash E'_{\mathrm{GL}(r,\mathbb{C})} = E_{\mathrm{GL}(r,\mathbb{C})}$, where $E_{\mathrm{GL}(r,\mathbb{C})}$ is the ramified principal $\mathrm{GL}(r,\mathbb{C})$–bundle corresponding to E_*, and $E'_{\mathrm{GL}(r,\mathbb{C})}$ is the principal $\mathrm{GL}(r,\mathbb{C})$–bundle corresponding to E'.

For any point $y \in Y$, let $\Gamma_y \subset \Gamma$ be the isotropy subgroup that fixes y for the action of Γ on Y. The action of Γ_y on the fiber of $\mathcal{O}_{\mathbb{P}(E')}(N(E_*)) := \mathcal{O}_{\mathbb{P}(E')}(1)^{\otimes N(E_*)}$ is trivial, where $N(E_*)$ is the integer in (3.4). Indeed, this follows immediately from the

construction of E_* for E'. Therefore, the quotient $\Gamma\backslash\mathcal{O}_{\mathbb{P}(E')}(N(E_*))$ defines a line bundle over $\Gamma\backslash\mathbb{P}(E') = \mathbb{P}(E_*)$. We have a natural isomorphism of line bundles

$$\Gamma\backslash\mathcal{O}_{\mathbb{P}(E')}(N(E_*)) \,=\, \mathcal{O}_{\mathbb{P}(E_*)}(1)\,. \tag{3.9}$$

Let

$$\psi_{E'}\,:\,\mathbb{P}(E')\,\longrightarrow\, Y$$

be the natural projection. For any $i\,\in\,[0\,,r]$, let

$$c_i(E')\,\in\,\mathrm{CH}^i(Y)_{\mathbb{Q}}\,:=\,\mathrm{CH}^i(Y)\otimes_{\mathbb{Z}}\mathbb{Q}$$

be the i–th Chern class of E'. We have

$$\sum_{i=0}^{r}\frac{(-1)^i}{N(E_*)^{r-i}}c_1(\mathcal{O}_{\mathbb{P}(E')}(N(E_*)))^{r-i}\psi_{E'}^*c_i(E')\,=\,\sum_{i=0}^{r}(-1)^i c_1(\mathcal{O}_{\mathbb{P}(E')}(1))^{r-i}\psi_{E'}^*c_i(E')\,=\,0 \tag{3.10}$$

(see [Ha, page 429]). The identity in (3.10) in fact uniquely determines the Chern classes of E' provided it is given that $c_0(E')\,=\,1$.

Since the vector bundle E' is Γ–linearized, it follows immediately that

$$c_i(E')\,\in\,(\mathrm{CH}^i(Y)_{\mathbb{Q}})^{\Gamma}\,, \tag{3.11}$$

where $(\mathrm{CH}^i(Y)_{\mathbb{Q}})^{\Gamma}$ is the invariant part of $\mathrm{CH}^i(Y)_{\mathbb{Q}}$ for the action of Γ on it. We also know that the pullback homomorphism

$$\gamma^*\,:\,\mathrm{CH}^i(X)_{\mathbb{Q}}\,\longrightarrow\,(\mathrm{CH}^i(Y)_{\mathbb{Q}})^{\Gamma}$$

is an isomorphism [Fu, pages 20–21, Example 1.7.6].

From (3.8) we have the quotient map

$$\beta\,:\,\mathbb{P}(E')\,\longrightarrow\,\mathbb{P}(E_*)$$

for the action of Γ, and from (3.9) it follows that

$$\beta^*\mathcal{O}_{\mathbb{P}(E_*)}(1)\,=\,\mathcal{O}_{\mathbb{P}(E')}(N(E_*))\,. \tag{3.12}$$

Hence

$$\beta^*c_1(\mathcal{O}_{\mathbb{P}(E_*)}(1))\,=\,c_1(\mathcal{O}_{\mathbb{P}(E')}(N(E_*)))\,.$$

Therefore, from (3.10) and (3.11) we conclude that for each $i\,\in\,[0\,,r]$, there is an unique element $\widetilde{C}_i\,\in\,\mathrm{CH}^i(X)_{\mathbb{Q}}$ such that $\widetilde{C}_0(E_*)\,=\,1/N(E_*)^r$ and

$$\sum_{i=0}^{r}(-1)^i c_1(\mathcal{O}_{\mathbb{P}(E_*)}(1))^{r-i}\psi^*\widetilde{C}_i(E_*)\,=\,0\,.$$

This completes the proof of the theorem. □

Definition 3.4. For any integer $i\,\in\,[0\,,r]$, the *i–th Chern class* $c_i(E_*)$ of a parabolic vector bundle E_* is defined to be

$$c_i(E_*)\,:=\,N(E_*)^{r-i}\cdot\widetilde{C}_i(E_*)\,\in\,\mathrm{CH}^i(X)_{\mathbb{Q}}\,,$$

where $\widetilde{C}_i(E_*)$ is the class in Theorem 3.3.

Corollary 3.5. *Let E_* be a parabolic vector bundle over X of rank r. Let $E' \longrightarrow Y$ be the corresponding Γ–linearized vector bundle (see the proof of Theorem 3.3). Then*

$$c_i(E') = \gamma^* c_i(E_*)$$

for all i.

Proof. From the construction of (3.7) using (3.10) it follows immediately that $\gamma^* \widetilde{C}_i(E_*) = c_i(E')/N(E_*)^{r-i}$. Therefore, the corollary follows from Definition 3.4. □

Define the *Chern polynomial* for E_* to be

$$c_t(E_*) = \sum_{i=0}^{r} c_i(E_*) t^i ,$$

where $r = \mathrm{rank}(E_*)$, and t is a formal variable. The *Chern character* of E_* is constructed from the Chern classes of E_* in the following way: if $c_t(E_*) = \prod_{i=1}^{r}(1+\alpha_i t)$, then

$$\mathrm{ch}(E_*) := \sum_{j=1}^{r} \exp(\alpha_j) \in \mathrm{CH}^*(X)_{\mathbb{Q}} .$$

Proposition 3.6. *Let E_* and F_* be parabolic vector bundles on X.*

1. *The Chern polynomial of the parabolic direct sum $E_* \oplus F_*$ satisfies the identity $c_t(E_* \oplus F_*) = c_t(E_*) \cdot c_t(F_*)$.*
2. *The Chern polynomial of the parabolic dual E_*^* satisfies the identity $c_t(E_*^*) = c_{-t}(E_*)$.*
3. *The Chern character of the parabolic tensor product $E_* \otimes F_*$ satisfies the identity $\mathrm{ch}(E_* \otimes F_*) = \mathrm{ch}(E_*) \cdot \mathrm{ch}(E_*)$.*

Proof. The Chern classes of usual vector bundles satisfy the above relations. The correspondence between the parabolic vector bundles and the Γ–linearized vector bundles takes the tensor product (respectively, direct sum) of any two Γ–linearized vector bundles to the parabolic tensor product (respectively, parabolic direct sum) of the corresponding parabolic vector bundles. Similarly, the dual of a given parabolic vector bundle corresponds to the dual of the Γ–linearized vector bundle corresponding to the given parabolic vector bundle. In view of these facts, the proposition follows form Corollary 3.5. □

4. Comparison with equivariant Chern classes

Let us recall the basic construction of equivariant intersection theory as in [EG]. Consider a smooth variety Z equipped with an action of a finite group G. Let V be a representation of G such that there is an open subset U of V on which G acts freely and the codimension of the complement $V - U$ is at least $\dim Z - i$. Following Edidin and Graham we write

$$Z_G = (Z \times U)/G .$$

The equivariant Chow groups are defined to be

$$A^i_G(Z) = A^i(Z_G) \otimes \mathbb{Q} .$$

It is shown in Proposition 1 of [EG] that this definition does not depend on V and U.

Consider a parabolic vector bundle E_* on X. Let $\gamma : Y \longrightarrow X$ be a Galois cover as in the proof of Theorem 3.3. The Galois group of γ will be denoted by G. Let E' be the G–linearized vector bundle on Y associated to E_*. The vector bundle E' has equivariant Chern classes

$$c_i(E') \in A^i_G(Y)\,.$$

We have a diagram

$$\begin{tikzcd}
\mathbb{P}(E_*) \arrow[d, "\psi_X"] & \mathbb{P}(E_*)_G = (\mathrm{Pr}(E_*) \times U)/G \arrow[l, "\pi_X"'] \arrow[d, "\psi^G_X"] & \mathbb{P}(E')_G \arrow[l, "\beta^G"'] \arrow[d, "\psi^G_Y"] \\
X & X_G = (X \times U)/G \arrow[l, "p_X"] & Y_G \arrow[l, "f_G"]
\end{tikzcd}$$

Note that the morphisms p_X and π_X are flat. Further, the morphisms β_G and f_G are flat and proper. The scheme $\mathbb{P}(E')_G$ is a projective bundle over Y_G as the action of G on $Y \times U$ is free; see also [EG, Lemma 1]. All these can be deduced by using the fact that the group G acts freely on $X \times U$ and $Y \times U$.

Proposition 4.1. *We have the following relationship amongst Chern classes:*

$$p^*_X(c_i(E_*)) = f_{G,*}(c^G_i(E'))\,.$$

Proof. By the projection formula it suffices to show that

$$f^*_G p^*_X(c_i(E_*)) = c^G_i(E')\,. \tag{4.1}$$

Flat pullback preserves intersection products so the equation obtained by pulling back (3.7) to $\mathbb{P}(E')$ remains valid. As was observed in the proof of Theorem 3.3 we have that

$$\beta^{G*}\pi^*_X(\mathcal{O}_{\mathbb{P}(E_*)}(1)) = \mathcal{O}_{\mathbb{P}(E')}(N(E_*))$$

(see (3.12)). Now using Definition 3.4 it is deduced that (4.1) holds. □

References

[BBN] V. Balaji, I. Biswas and D. S. Nagaraj, Ramified G–bundles as parabolic bundles, *Jour. Ramanujan Math. Soc.* **18** (2003), 123–138.

[Bi1] I. Biswas, Parabolic bundles as orbifold bundles, *Duke Math. Jour.* **88** (1997), 305–325.

[Bi2] I. Biswas, Chern classes for parabolic bundles, *Jour. Math. Kyoto Univ.* **37** (1997), 597–613.

[Bi3] I. Biswas, Connections on a parabolic principal bundle, Canadian Jour. Math. **58** (2006), 262–281.

[Bi4] I. Biswas, Connections on a parabolic principal bundle, II, *Canadian Math. Bull.* **52** (2009), 175–185.

[BD] I. Biswas and A. Dhillon, Semistability criterion for parabolic vector bundles on curves, *Jour. Math. Sci. Univ. Tokyo* **18** (2011), 181–191.

[Bo1] N. Borne, Fibrés paraboliques et champ des racines, *Int. Math. Res. Not.* (2007), no. 16, Art. ID rnm049.

[Bo2] N. Borne, Sur les représentations du groupe fondamental d'une variété priveé d'un diviseur à croisements normaux simples, *Indiana Univ. Math. Jour.* **58** (2009), 137–180.

[EG] D. Edidin and W. Graham, Equivariant intersection theory, *Invent. Math.* **131** (1998), 595–634.

[Fu] W. Fulton, *Intersection Theory*, Second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, Folge 3, Vol. 2, Springer-Verlag, Berlin, 1998.

[Ha] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977.

[IS] J. N. Iyer and C. T. Simpson, A relation between the parabolic Chern characters of the de Rham bundles, *Math. Ann.* **338** (2007), 347–383.

[MY] M. Maruyama and K. Yokogawa, Moduli of parabolic stable sheaves, *Math. Ann.* **293** (1992), 77–99.

[Se] C. S. Seshadri, Moduli of vector bundles on curves with parabolic structures, *Bull. Amer. Math. Soc.* **83** (1977), 124–126.

[Ta] C. H. Taher, Calculating the parabolic Chern character of a locally abelian parabolic bundle, *Manuscr. Math.* **132** (2010), 169–198.

[Yo] K. Yokogawa, Infinitesimal deformations of parabolic Higgs sheaves, *Int. Jour. Math.* **6** (1995), 125–148.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: indranil@math.tifr.res.in

Department of Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada

E-mail address: adhill3@uwo.ca